RELATIVE ENTROPY IN FIELD THEORY, THE H THEOREM AND THE RENORMALIZATION GROUP

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We consider relative entropy in Field Theory as a well defined (non-divergent) quantity of interest. We establish a monotonicity property with respect to the couplings in the theory. As a consequence, the relative entropy in a field theory with a hierarchy of renormalization group fixed points ranks the fixed points in decreasing order of criticality. We argue from a generalized H theorem that Wilsonian RG flows induce an increase in entropy and propose the relative entropy as the natural quantity which increases from one fixed point to another in more than two dimensions.

1 Introduction

This work is motivated by the old idea of irreversibility in the renormalization group (RG). Specifically, the existence of a function monotonic with the RG (a Lyapunov function for it) was proposed to ensure the existence of regular fixed points as opposed to other possible pathological behavior, such as limit cycles.[1] As a sufficient condition for the existence of that function, one might have that the RG is actually a gradient dynamical system, that is, it derives from a potential, which is then the monotonic function. This type of RG flow would have further convenient properties. For example, it implies that the matrix that represents the linear RG near a fixed point (FP) is symmetric and hence it has real eigenvalues, to be identified with anomalous dimensions. Despite evidence—only perturbative and at low order—for the gradient flow,[1] its existence remains inconclusive.

However, in two dimensions a monotonic function has been found with the methods of conformal field theory, Zamolodchikov's C function. There have been many attempts to generalize the C theorem to higher dimensions but they cannot be considered definitive.[2,3,4] Moreover, these authors have ignored Zamolodchikov's initial motivation, the adaptation of the renowned Boltzmann's H theorem to the RG setting. In a synthetic formulation we may quote this theorem as the statement that non-equilibrium (coarse-grained) entropy increases with time in the approach to equilibrium. Kadanoff block transformations as well as other coarse-graining formulations of the RG discard short-range information on a physical system and hence seem irreversible in a similar sense to the situations described by the H theorem, where the physical time is replaced by the renormalization group parameter.

In this spirit, we define a field theoretic entropy and study its properties, in particular, its possible divergences and its monotinicity. In a phase diagram with a multicritical point the entropy relative to it is finite and a monotonic function of the coupling constants spanning that diagram measured from that point. Therefore, this entropy is monotonic in the crossover from this point to another more stable multicritical point. It can be realized by a RG flow in which one or several relevant coupling constants become irrelevant. This is the typical situation that arises in field theoretic formulations of the RG. However, in coarse-grained or Wilson formulations the number of coupling constants is very large, as a consequence of the action of the RG itself, and they are classified as relevant or irrelevant only near the end fixed point. In analogy with the statistical mechanics of irreversible processes, we can then invoke a very general formulation of the H theorem due to E.T. Jaynes to show that the entropy increases in this more general context as well.

This account is based on previous work on this subject.[5]

2 Field theory entropy. Definition and properties

Let us begin by recalling some concepts of probability theory. Given a probability distribution $\{p_m\}$ one can define an entropy

$$S(\{p_m\}) = -\sum_{m=1}^{M} p_m \log p_m. \tag{1}$$

For a continous distribution

$$S[p(x)] = -\int dx\, p(x) \log p(x). \tag{2}$$

In this case is more a useful to define relative entropy with reference to a given distribution $q(x)$

$$S[p(x)] = \int dx\, p(x) \log \frac{p(x)}{q(x)}. \tag{3}$$

If $q(x)$ is the uniform distribution this definition coincides up to a conventional sign and a constant with the previous one, now called the absolute entropy. However, it is more correct to take as reference a normalizable distribution, for example, the Gaussian distribution.

Now we are ready to define an entropy in field theory. We begin with the usual definition of the generating functional or partition function

$$Z[\{\lambda\}] = e^{-W[\{\lambda\}]} = \int \mathcal{D}[\phi]\, \mathrm{e}^{-I[\phi,\{\lambda\}]} \tag{4}$$

with $W[\{\lambda\}]$ the generator of connected amplitudes. If we write

$$\int \mathcal{D}[\phi]\, \mathrm{e}^{-I[\phi,\{\lambda\}]+W[\{\lambda\}]} = 1, \tag{5}$$

we can consider

$$\mathcal{P}[\phi,\{\lambda\}] = \mathrm{e}^{-I[\phi,\{\lambda\}]+W[\{\lambda\}]}$$

as a (family of) probability distribution of the stochastic variables ϕ. Therefore, the relative entropy is

$$\begin{aligned} \mathcal{S}[\mathcal{P},\mathcal{P}_0] &= \mathrm{Tr}[\mathcal{P}\ln(\mathcal{P}/\mathcal{P}_0)] \\ = \int \mathcal{D}[\phi]\,(-I+I_0+W-W_0)\, e^{-I+W} &= W - W_0 - \langle I - I_0\rangle \end{aligned} \tag{6}$$

with I_0 a reference action of the same family, that is, with given values of the coupling constants.

The typical form of the action is a linear combination of composite fields with their respective coupling constants,

$$I[\phi,\{\lambda\}] = \lambda^a f_a[\phi]. \tag{7}$$

We assume that only a part of the action I_r is of interest to us. (We shall call it later relevant o crossover action for physical reasons.) We introduce a global coupling constant z for it or, in other words, we extract a common factor from the coupling constants in it,

$$z\, I_r = I - I_0. \tag{8}$$

Then we can express the relative entropy as the Legendre transform of $W - W_0$ with respect to z:

$$\mathcal{S}_{\mathrm{rel}} = W(z) - W(0) - z\frac{dW}{dz}. \tag{9}$$

Alternatively, it is the Legendre transform with respect to the couplings in I_r, the relevant couplings l_a,

$$\mathcal{S}_{\mathrm{rel}} = W - W_0 - l^a \partial_a W. \tag{10}$$

In turn the absolute entropy is the Legendre transform with respect to all the couplings. Both entropies have a simple description in perturbation theory: They are sums of n particle irreducible (nPI) diagrams with external composite fields at zero momentum.

We have already remarked that for continuous probability distributions the relative entropy is well defined whereas the absolute entropy is not. In Field

Theory, however, the stochastic variable is a field or, in other words, there is an infinity of stochastic variables that causes that the previous quantities are not necessarily well defined. This is the well known problem of divergences, either ultraviolet (UV) or infrared (IR). If we assume that the field theory under study is regularized by defining it on a finite lattice, we return to the case of a finite number of (continuous) stochastic variables and W as well as the relative entropy are well defined. Then the first limit we have to consider is the infinite volume or thermodynamic limit, as is usual in Statistical Mechanics. To perform the limit one has to define specific quantities—per lattice site or unit volume—that remain finite. Nevertheless, at the critical point some of them may diverge, as occurs to the specific heat. We speak of an IR divergence. In (6) or (10) the possible IR divergences of the relative entropy are in the sum of $\langle f_a \rangle$ and hence it is IR finite in $d > 2$.

The second and more important limit is the continuum limit, which allows to define a field theory as the number of lattice points per unit volume goes to infinity. Since W per lattice site is finite, the relevant quantity now, W per unit volume, diverges as a^{-d}, with a the lattice spacing and d the dimension. This is a UV divergence, closely related with the divergent ground state energy in Quantum Field Theory, which is substracted by taking the composite operators normal ordered. Fortunately, the a^{-d} divergence of W is cancelled in the Legendre transform (9) that gives the relative entropy by substracting $W(0)$. Furthermore, the next-to-leading divergence a^{-d+2} cancels in the substraction of $z\frac{dW}{dz}$. Therefore, in dimension $d < 4$ the relative entropy is UV finite. Notwithstanding this, we know that if we calculate nPI diagrams in perturbation theory we shall find UV divergences. These divergences pertain exclusively to perturbation theory and are removed by renormalization of the coupling constants. Quantities expressed in terms of physicial (renormalized) couplings show no trace of perturbative divergences.

2.1 Monotonicity of the relative entropy

From the expression of the relative entropy as a Legendre transform (9) we obtain

$$\frac{d\mathcal{S}}{dz} = \frac{dW}{dz} - \frac{d}{dz}\left(z\frac{dW}{dz}\right) = -z\frac{d^2W}{dz^2} = -z\frac{d}{dz}\langle I_r \rangle = -z(-\langle I_r^2 \rangle + \langle I_r \rangle^2). \quad (11)$$

Then we have the positive quantity

$$z\frac{d\mathcal{S}}{dz} = z^2\langle (I_r - \langle I_r \rangle)^2 \rangle \geq 0, \quad (12)$$

which proves that the relative entropy is a monotonic function of $|z|$. Moreover,

$$\langle (I_r - \langle I_r \rangle)^2 \rangle = l^a \langle (f_a - \langle f_a \rangle)(f_b - \langle f_b \rangle) \rangle \, l^b. \tag{13}$$

Hence the matrix

$$Q_{ab} = \langle (f_a - \langle f_a \rangle)(f_b - \langle f_b \rangle) \rangle = -\frac{\partial^2 W}{\partial l^a \partial l^b} \tag{14}$$

is positive definite, implying that $W(l^a)$ is convex. This matrix is generally defined in probability theory, where it is called the Fisher information matrix and provides a metric in the space of distributions.

There is a similar convexity property for the relative entropy as function of its natural variables, the expectation values of composite fields, $\mathcal{S}(\langle f_a \rangle)$; since

$$\frac{\partial^2 \mathcal{S}}{\partial \langle f_a \rangle \partial \langle f_b \rangle} = -\frac{\partial l^a}{\partial \langle f_b \rangle} = -\left(\frac{\partial \langle f_b \rangle}{\partial l^a} \right)^{-1} = -\left(\frac{\partial^2 W}{\partial l^a \partial l^b} \right)^{-1} = Q_{ab}^{-1}, \tag{15}$$

the matrix of second derivatives of the entropy is the inverse of Q and hence positive definite as well. Summarizing, the key properties of the entropy are its monotonicity with z, therefore, any coupling, and its convexity in the space of composite fields $\langle f_a \rangle$.

3 Crossover between field theories

The situation we consider is when the action I is associated with some multicritical point and there is crossover to another lower multicritical point. Crossover signifies a change of the type of critical behaviour or universality class, with a change of critical exponents or any other quantity pertaining to it. Then I_0 refers to the critical action at the first point and I_r contains the relevant coupling constants. The way in which a crossover occurs is illustrated by the action of the RG: Its flow will lead away from the first point asymptotically to a region of the phase diagram of lower dimensionality, where one or more couplings become irrelevant and dissapear.

We will restrict our considerations in what follows to scalar $\mathbb{Z}_2$ symmetric field theories in $2 < d < 4$ with polynomial potentials and non symmetry breaking fields. For illustration, we will discuss some exact results pertaining to soluble statistical models, which illuminate the behaviour of the field theories in the same universality classes.

3.1 The Gaussian model and the zero to infinite mass crossover

As an elementary example we take the Gaussian model, given by the action

$$I[\phi] = \int_{\mathcal{M}} \left\{ \frac{1}{2}(\partial\phi)^2 + \frac{r}{2}\phi^2 \right\} . \tag{16}$$

Here the relevant coupling is r. Its critical value is $r_c = 0$. Hence the crossover parameter is also r and the end fixed point for large r is the infinite mass Gaussian model.

The Gaussian model is soluble in any dimension by direct functional integration. The lattice version with coupling $\beta = 1/T$ yields [7]

$$W[\beta] = \frac{1}{2} \int_{-\pi}^{\pi} \frac{d^d k}{(2\pi)^d} \ln \left\{ 1 - 2\beta \sum_i^d \cos k_i \right\} \tag{17}$$

per site. The continuum limit is performed by redefining momenta as $k = ap$ and considering W per unit volume. Since k belongs to a Brillouin zone, $-\infty < p < \infty$. It is straight forward to check that the UV divergences are as announced in the previous section and cancel in the relative entropy.

3.2 Ising universality class

Let us next consider the Ising model on a rectangular lattice. For simplicity we will restrict our considerations to equal couplings in the different directions. Since the random variables here (the Ising spins) take discrete values it is natural to consider the absolute entropy. This model, as is well known, admits an exact solution in two dimensions for the partition function [8] which yields a W per site

$$W[\beta] = \frac{1}{2} \int_{-\pi}^{\pi} \frac{d^2 k}{(2\pi)^2} \ln \left[\cosh^2(2\beta) - \sinh(2\beta)(\cos k_x + \cos k_y) \right] . \tag{18}$$

The entropy is then

$$S_a = -\left(W(\beta) - \beta \frac{dW(\beta)}{d\beta} \right)$$

as plotted against β in figure 1a. The monotonicity of the entropy turns to convexity when it is expressed in terms of the internal energy $U = \frac{dW}{d\beta}$ as can be seen in figure 1b.

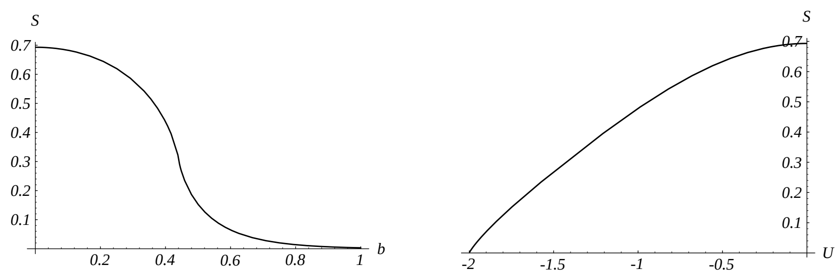

Fig. 1a and 1b: The entropies $S_a(\beta)$ and $S_a(U)$ for the 2d Ising model

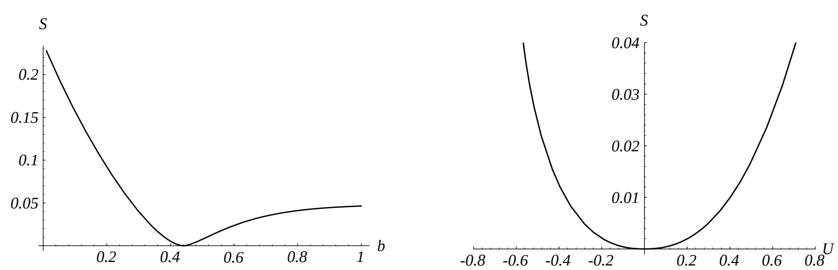

Fig. 2a and 2b: The relative entropies $S(\beta, \beta^*)$ and $S(U, U^*)$ for the 2d Ising model

Now, of course, we can also consider relative entropy in this setting. To facilitate comparison with a field theory it is natural to choose entropy relative to the critical point (CP) lattice Ising model. This is also natural since the critical point is a preferred point in the model. This relative entropy is given

by

$$S = W(\beta) - W(\beta^*) - (\beta - \beta^*)\frac{dW(\beta)}{d\beta}$$

where $\beta^* = \frac{1}{2}\ln(\sqrt{2}+1) \sim 0.4406868$ is the critical coupling of the Ising model. We have plotted this in figure 2a. We see that it is a monotonic increasing function of $|\beta - \beta^*|$ and is zero at the critical point. In figure 2b we plot this entropy as a function of the relevant expectation value, the internal energy $U = \frac{dW}{d\beta}$, and set the origin at U^*, the internal energy at the critical point. Naturally, the graph is convex.

In the continuum limit the absolute entropy or W per unit volume are UV divergent but the relative entropy is finite, like in the Gaussian model. In fact, W is similar in both models when expressed in terms of the mass of the particles, although the relation of the mass with the coupling constant is very different in each case, of course.

In more than two dimensions the Ising model has not been solved exactly. Its critical behaviour is in the universality class of a ϕ^4 field theory,

$$I[\phi] = \int_{\mathcal{M}} \left\{ \frac{\alpha}{2}(\partial\phi)^2 + \frac{r}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 \right\}. \tag{19}$$

We restrict our considerations to $d < 4$ where the theory is super-renormalizable. The CP occurs for r_c such that the correlation length is infinite, with some arbitrary but fixed value of the bare coupling constant λ. These are the reference values for the relative entropy. Since the squared mass r is a function of the Ising coupling β, going off from β_c induces crossover: This is the crossover line from the Wilson Fisher fixed point to the infinite mass Gaussian fixed point. Now we can apply the previous monotonicity theorem with $z = r - r_c$. Since λ is relevant in $d < 4$, one can place it into the crossover portion of the action. This provides us with another crossover and in this more complicated phase diagram there are in fact two Gaussian fixed points: A massless and an infinite mass one, both associated with $\lambda = 0$ (see the article by Nicoll et al [6] for a description of the total phase diagram). The crossover between them is that associated with the Gaussian model as described above.

3.3 The universality class of the tricritical model

We consider now models with a tricritical point (TCP), namely, with a phase diagram where three critical lines meet at a point and three phases become critical simultaneously. The universality class of those models is a field theory

with a sixth degree coupling, with action

$$I[\phi, \{\lambda(2)\}] = \int_{\mathcal{M}} \left\{ \frac{1}{2}(\partial\phi)^2 + \frac{r}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 + \frac{g}{6!}\phi^6 \right\} \tag{20}$$

This field theory is superrenormalizable for $d < 3$. The TCP occurs for r_{tc} and λ_{tc} such that the correlation lengths of the two order parameters ϕ and ϕ^2 are infinite, with some arbitrary but fixed value of the bare coupling constant g. These are the reference values for the relative entropy.

Now we have various possible crossovers. If g is held fixed the relevant variables are $t = r - r_{tc}$ and $l = \lambda - \lambda_{tc}$. First consider the line formed setting $l = 0$ and ranging t from zero to infinity. This is a line leaving the tricritical point and going to an infinite mass Gaussian model. Similarly we can consider the line $t = 0$ and l ranging through different values. If λ is held at its critical value one sweeps a critical line and crosses over to the Wilson-Fisher fixed point. According to the monotonicity theorem we have again that the relative entropy is a monotonic function in these crossovers.

A more thorough analysis of the structure of the phase diagram pertinent to these crossovers can be done in terms of the shape of RG trajectories. They can be easily obtained near the TCP from the linear RG equations as $t = c\, l^{\varphi}$ for various c, with only one parameter given by the ratio of scaling dimensions of the relevant fields $\varphi = \frac{\Delta_t}{\Delta_l} > 1$, called the crossover exponent. These curves have the property that they are all tangent to the t axis at the origin and any straight line $t = a\, l$ intersects them at some finite point, $l_i = (a/c)^{\frac{1}{\varphi - 1}}$ and $t_i = a l_i$. For any given c the values of l_i and t_i increase as a decreases and go to infinity as $a \to 0$. This clearly shows that the stable fixed point of the flow is on the line at infinity and, in particular, its projective coordinate is $a = 0$. The point $a = \infty$ on the line at infinity is also fixed but unstable. In general, as the overall factor z is taken to infinity we shall hit some point on the separatrix connecting these two points at infinity.

We recognize here a feature of the type of crossover phase diagrams that we study: In a coordinate system defined by the linear RG (usually called system of nonlinear scaling fields) near the highest multicritical point the remaining fixed points are located at infinity. To study the crossover, when a FP is at infinity, we need to perform some kind of compactification of the phase diagram. Thus, we shall think of the total phase diagram as a compact manifold containing the maximum number of *generic* RG FP. This point of view is especially sensible regarding the topological nature of RG flows. Furthermore, thinking of the RG as just an ODE indicates what type of compactification of phase diagrams is adequate: It is known in the theory of ODEs that the analysis of the flow at infinity and its possible singularities can be done by

completing the affine space to projective space.[9] This is also appropriate for phase diagrams.

The projective compactification lends itself to an interesting physical interpretation. In the tricritical action, for example, we can consider the set of couplings r, λ and g as a system of homogeneous coordinates in the real projective space $\mathbb{R}P_2$. Holding g fixed is equivalent to taking the patch of $\mathbb{R}P_2$ appropriate to the TCP, with affine coordinates r/g and λ/g. (We may set $g = 1$, say, where we now use dimensionless couplings, the original g, which we now label g_B, setting the scale.) Following the RG trajectories, in the IR limit g_R becomes independent of its initial value and eventually goes to zero, as illustrated by the solution of the one-loop RG equations,

$$g_R = \frac{g_B}{1 + a(d)\, g_B\, R^{3-d}}$$

with R the IR cutoff and $a(d)$ a dimension dependent factor. This is the physical reason why the other FP are at infinite distance of the TCP and one must take a coordinate system with $g = 0$ or, in mathematical terms, choose another coordinate patch in $\mathbb{R}P_2$.

4 Wilson's RG and H theorem

A Wilson RG transformation is such that it eliminates degrees of freedom of short wave length and hence high energy. Typical examples are decimation or block spin transformations. It is intuitively clear that their action discards information on the system and therefore must produce an increase of entropy. Indeed, as remarked by Ma [10] iterating this type transformation does not constitute a group but rather a semi-group, since the process cannot be uniquely reversed. In the language of statistical mechanics we can think of it as an irreversible process.

We illustrate Wilson's RG by a very simple example, the Gaussian model of subsection 3.1 with cut-off action

$$I = \frac{1}{2}\int_0^\Lambda d^d p\ \phi(p)\ \left(p^2 + r\right)\ \phi(-p), \tag{21}$$

which yields

$$W[z] = \frac{1}{2}\int_0^\Lambda \frac{d^d p}{(2\pi)^d}\ \ln\frac{p^2 + r}{\Lambda^2}. \tag{22}$$

The corresponding relative entropy

$$\mathcal{S}[z] = \frac{1}{2}\int_0^\Lambda \frac{d^d p}{(2\pi)^d}\left(\ln\frac{p^2 + r}{p^2 + r_c} - \frac{t}{p^2 + r}\right) \tag{23}$$

is finite when Λ goes to infinity and vanishes for $t = 0$. The differential Wilson RG is implemented by letting Λ run to lower values. Let us see that S is monotonic with Λ.

We have that

$$\frac{\partial \mathcal{S}}{\partial \Lambda} = \frac{\Lambda^{d-1}}{2^d \pi^{\frac{d}{2}}\, \Gamma(d/2)} \left(\ln \frac{\Lambda^2 + r}{\Lambda^2 + r_c} - \frac{t}{\Lambda^2 + r} \right), \tag{24}$$

except for an irrelevant constant. With the change of variable $x = \Lambda^2$, we have to show that the corresponding function of x is of the same sign everywhere. Then we want

$$\ln \frac{x + r}{x + r_c} - \frac{r - r_c}{x + r}$$

not to change sign. Interestingly, the properties of this expression are independent of x somehow for if one substitutes in $\ln \rho - \frac{\rho - 1}{\rho}$ the value $\rho = \frac{x+r}{x+r_c}$ then one recovers the entire function. Now it is easy to show that $\ln \rho \geq 1 - \frac{1}{\rho}$. (The equality holds for $\rho = 1$—the critical point.)

This proof resembles the classical proofs of H theorems. Boltzmann's H theorem ($H = -S$ of (1)) states that the coarse-grained entropy of a rarefied gas, described by its one-particle statistical distribution $f(\boldsymbol{r}, \boldsymbol{p})$, increases as the gas evolves to its Maxwell-Boltzmann equilibrium distribution, where it remains constant, effectively making this evolution an irreversible process.[11] His proof uses the expression for $\frac{df}{dt}$ given by the kinetic equation that takes into account collisions between particles. However, the negativity of $\frac{dH}{dt}$ is ultimately due to the positivity of the function $(x-1)\ln x$, in a similar fashion to the proof in the previous paragraph.

For general far-from-equilibrium processes a description in terms of differential equations (with respect to time) is not available. A typical irreversible process is, for example, the one that takes place when the partition which thermally insulates two parts of a container with different temperature is suddenly removed. The container (the total system) passes through states which cannot be regarded as thermodynamical states, that is, described by a small number of variables, until it reaches a new equilibrium state where the temperature is well defined and any trace of the initial values in the two parts is lost. The important point is that, since the entropy is a state variable, its increase is independent of the details of the process and one can assume a quasi-static process for convenience.

We can think of the action of Wilson's RG in this manner: typically its action introduces new couplings and eventually their number is not bound and the description in terms of the original Hamiltonian is no longer useful.

However, when a FP is approached the number of couplings reduces again, in fact to a smaller number than the initial one since some information is necessarily lost: A relevant coupling must become irrelevant in the process. Here too it may be convenient a description of the process, a crossover, by quasi-equilibrium states, such as is provided by the Field Theory picture presented in the previous section. However, to independently establish the increase of entropy we can ressort to the very general formultion of the H theorem provided by Jaynes in the form of what has been called the maximum entropy principle.[12]

In Jayne's maximum entropy principle H is a function(al) of the probability distribution of the system that measures the information available to the system and has to be a minimum at equilibrium. To be precise, the actual probability distribution is such that it does not contain information other than that implied by the constraints or boundary conditions imposed at the outset. The simplest case of the H theorem is when there is no constraint wherein H is a minimum for a uniform distribution. This is sometimes called the principle of equiprobability. This is the case for an isolated system in statistical mechanics: all the states of a given energy have the same probability (micro-canonical distribution). Another illustrative example is provided by a system thermally coupled to a heat reservoir at a given temperature where we want to impose that the average energy takes a particular value. Minimizing H then yields the canonical distribution.

In general, we may impose constraints on a system with states X_i that the average values of a set of functions of its state, $f_r(X_i)$, adopt pre-determined values,

$$\langle f_r \rangle := \sum_i P_i \, f_r(X_i) = \bar{f}_r, \tag{25}$$

with $P_i := P(X_i)$. The maximum entropy formalism leads to the probability distribution [13]

$$P_i = Z^{-1} \exp\left(- \sum_r \lambda_r \, f_r(X_i) \right). \tag{26}$$

The λ_r are Lagrange multipliers determined in terms of $\bar{f}_r$ through the constraints. In field theory a state is defined as a field configuration $\phi(x)$. One can define functionals of the field $\mathcal{F}_r[\phi(x)]$. These functionals are usually quasi-local and are called composite fields. The physical input of a theory can be given in two ways, either by specifying the microscopic couplings or by specifying the expectation values of some composite fields, $\langle \mathcal{F}_r[\phi(x)] \rangle$. The maximum

entropy condition provides an expression for the probability distribution,

$$P[\phi(x)] = Z^{-1} \exp\left(- \sum_r \lambda_r \, \mathcal{F}_r[\phi(x)] \right), \tag{27}$$

and therefore for the action,

$$I = \sum_r \lambda_r \, \mathcal{F}_r; \tag{28}$$

namely, a linear combination of relevant fields with coupling constants to be determined from the specified $\langle \mathcal{F}_r \rangle$. If a constraint is released, namely, the information given by the expectation value of a composite field is lost, the system evolves—crosses over—to a more stable state with larger entropy. Its probability distribution is again of the form (27) but with one less term in the sum, the irrelevant field that has dissapeared.

5 Conclusions

We have shown that the relative entropy is the natural definition of entropy in Field Theory; namely, this relative entropy is well defined, that is, free from divergences, as opposed to the absolute entropy. It is then the appropriate quantity to study crossover between field theories.

We have established a theorem of monotonicity of the relative entropy with respect to the coupling constants. Thus it provides a natural function which ranks the different critical points in a model. It grows as one descends the hierarchy in the crossovers between scalar field theories corresponding to different multicritical points.

We have further established that the phase diagrams of the hierarchy of critical points are associated with a nested sequence of projective spaces. It is convenient to use coordinates adapted to a particular phase diagram in the hierarchy. Hence a crossover implies a coordinate change. The transition from bare to renormalized coordinates provides a method of compactifying the phase diagram.

We discussed the action of the Wilson RG and argued that the relative entropy increases as more degrees of freedom are integrated out. Although a differential increase may be hard to get in general due to the proliferation of couplings, Jaynes' formulation of the H theorem allows us to conclude that the entropy increases globally in a crossover.

This study was motivated by the search for a monotonic function *along* the RG trajectories, like the C function given by Zamolodchikov's theorem in

two dimensions. Thus we owe a comment to the connection with this theorem. The field theoretic relative entropy is a well defined and useful quantity on its own, as we have shown. Since we restricted the dimension to $d > 2$ a direct comparison with the C function is not possible. Given that the C function is built from correlation data, unlike the relative entropy, they do not seem related. However, recent work [14] relates the central charge with a particular type of entropy, the geometrical entropy. Undoubtedly, this connection needs to be further investigated.

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